END Scheme

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = b$$

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \qquad x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$$

$$\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

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cell average : $\bar{v}_i \equiv \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(\xi)\, d\xi$

k-th order accurate polynomial : $p_i(x) = v(x) + O(\Delta x^k)$

$$\therefore v^-_{i+\frac{1}{2}} = p_i(x_{i+\frac{1}{2}}) \qquad v^+_{i-\frac{1}{2}} = p_i(x_{i-\frac{1}{2}})$$

both are k-th order accurate.

stencil : $S(i) = \{I_{i-r}, \cdots, I_{i+s}\}$, $r+s+1=k$

$$\boxed{\begin{array}{l}\exists \text{ a unique } k-1 \text{ order polynomial such that} \\ \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_i(\xi)\, d\xi = \bar{v}_j, \quad j = i-r, \cdots, i+s\end{array}}$$

$$v^-_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j} \qquad v^+_{i-\frac{1}{2}} = \sum_{j=0}^{k-1} \tilde{c}_{rj} \bar{v}_{i-r+j}$$

for fixed $S(i)$, the value of $v^-_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}}$.

so,

$$\tilde{c}_{rj} = c_{r-1,j}$$

$$\therefore v_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} \bar{v}_{i-r+j}$$

ENO Scheme

let $V(x) = \int_{-\infty}^{x} v(\xi)\,d\xi$

$$V(x_{i+\frac{1}{2}}) = \sum_{j=-\infty}^{i} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(\xi)\,d\xi = \sum_{j=-\infty}^{i} \bar{v}_j \Delta x_j$$

let $P(x)$ be the unique polynomial that interpolates $V(x)$ at $x_{i-r+\frac{1}{2}}, \cdots, x_{i+s+\frac{1}{2}}$

and
$$p(x) \equiv P'(x)$$

$$\therefore \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p(\xi)\,d\xi = \bar{v}_j \quad , j = i-r, \cdots, i+s$$

$P(x)$ can be assumed in the form of Lagrange polynomial

$$P(x) = \sum_{m=0}^{k} V(x_{i-r+m-\frac{1}{2}}) \prod_{\substack{l=0 \\ l \neq m}}^{k} \frac{x - x_{i-r+l-\frac{1}{2}}}{x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}}$$

note that $\displaystyle\sum_{m=0}^{k} \prod_{\substack{l=0 \\ l \neq m}}^{k} \frac{x - x_{i-r+l-\frac{1}{2}}}{x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}} = 1$

$$\therefore P(x) = V(x_{i-r-\frac{1}{2}}) + \sum_{m=0}^{k} \left(V(x_{i-r+m-\frac{1}{2}}) - V(x_{i-r-\frac{1}{2}}) \right) \prod_{\substack{l=0 \\ l \neq m}}^{k} \frac{x - x_{i-r+l-\frac{1}{2}}}{x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}}}$$

Taking derivative,

ENO Scheme

$$p(x) = \sum_{m=0}^{k} \sum_{j=0}^{m-1} \bar{v}_{i-r+j} \Delta x_{i-r+j} \left[\frac{\sum_{\substack{l=0 \\ l\neq m}}^{k} \prod_{\substack{q=0 \\ q\neq m,l}}^{k} (x - x_{i-r+q-\frac{1}{2}})}{\prod_{\substack{l=0 \\ l\neq m}}^{k} (x_{i-r+m-\frac{1}{2}} - x_{i-r+l-\frac{1}{2}})} \right]$$

Approximate $\frac{1}{\Delta x_i}(\hat{v}_{i+\frac{1}{2}} - \hat{v}_{i-\frac{1}{2}}) = v'(x_i) + O(\Delta x^k)$

$\hat{v}_{i+\frac{1}{2}} \equiv \hat{v}(v_{i-r}, \dots, v_{i+s})$ $\quad i = 0, 1, \dots, N$

$v_i \equiv v(x_i)$

Assume that grid is uniform.

let $v(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi) d\xi$

$\therefore v'(x) = \frac{1}{\Delta x}\left[h(x+\frac{\Delta x}{2}) - h(x-\frac{\Delta x}{2})\right]$

$\hat{v}(x_{i+\frac{1}{2}}) = \hat{v}_{i+\frac{1}{2}} = h(x_{i+\frac{1}{2}}) + O(\Delta x^k)$

As what was done above, let $H(x) = \int_{-\infty}^{x} h(\xi) d\xi$

$\therefore \hat{v}_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} c_{rj} v_{i-r+j}$

ENO Approximation

Newton divided difference :

$V[x_{i-\frac{1}{2}}] = V(x_{i-\frac{1}{2}})$

$$V[x_{i-\frac{1}{2}}, \dots, x_{i+j-\frac{1}{2}}] = \frac{V[x_{i+\frac{1}{2}}, \dots, x_{i+j-\frac{1}{2}}] - V[x_{i-\frac{1}{2}}, \dots, x_{i+j-\frac{3}{2}}]}{x_{i+j-\frac{1}{2}} - x_{i-\frac{1}{2}}}$$

ENO Scheme

$$\cancel{P_k(x) = f_0(x_0) + f(x_1)(x-x_0) + f(x_1)(x-x_0)(x-x_1) + \cdots + f(x_k)(x-x_0)\cdots(x-x_{k-1})}$$

$\cancel{f(x_k) =}$

Let $P_k(x)$ be a k^{th} order polynomial, interpolating $f(x)$ at points $x_0, x_1, \cdots, x_k$

$$P_k(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \cdots + a_k(x-x_0)\cdots(x-x_{k-1})$$

$$a_0 = f[x_0], \quad a_1 = f[x_0, x_1], \quad \cdots, \quad a_k = f[x_0, \cdots, x_k]$$

$$f[x_0, \cdots, x_i] = \frac{f[x_1, \cdots x_i] - f[x_0, \cdots, x_{i-1}]}{x_i - x_0}$$

proof: (by induction) Let $q(x)$ be $k-1^{th}$ order polynomial interpolating $x_1, \cdots, x_k$.

$$P_{k-1}(x) = P_{k-2}(x) + f[x_0, \cdots, x_{k-1}](x-x_0)\cdots(x-x_{k-2})$$
$$= f[x_0, \cdots, x_{k-1}]x^{k-1} + \cdots$$

$$\therefore q(x) = f[x_1, \cdots, x_k]x^{k-1} + \cdots$$

$$\therefore P_k(x) = q(x) + \frac{x - x_k}{x_k - x_0}[P_{k-1}(x) - q(x)]$$
$$= \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0} x^k + \cdots$$

$\square$